Jacobians over $\mathbb{C}$

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What follows is a very quick jog through half of Mumford's "Curves and their Jacobians", Chapter III.

Mumford begins:

> *I would like to being by introducing Jacobians in the way that they were discovered historically. Unfortunately, my knowledge of 19th-century literature is very scant so this should not be taken literally.*

In my case, I need to add a further disclaimer:

> *Never mind the 19th century, I have a hard enough time with my own century already... So none of this should be taken literally. Or seriously, for that matter.*

Integration on $\mathcal{C}$

Historically: long-dead folks studied algebraic integrals $I = \int f(x)dx$, where $F(x, y = f(x)) = 0$.

So let's look at integrals of rational differentials on an algebraic curve $\mathcal{C}$:

$$I(a) = \int_{a_0}^{a} \omega \quad \text{where} \quad \omega = \frac{P(x,y)}{Q(x,y)}dx$$

with P, Q polynomials, a, a_0 in $\mathcal{C} : F(x, y) = 0$.

Abel's theorem

The main result is an addition theorem:
Let ω be a differential on $\mathcal{C}$. There exists an integer g such that if

- a_0 is a base point and
- $a_1, \ldots, a_{g+1}$ are any points on $\mathcal{C} \setminus \{\text{poles of } \omega\}$,
- then we can determine $\{b_1, \ldots, b_g\} \subset \mathcal{C} \setminus \{\text{poles of } \omega\}$ rationally in terms of the a_i such that

$$\int_{a_0}^{a_1} \omega + \cdots + \int_{a_0}^{a_{g+1}} \omega = \int_{a_0}^{b_1} \omega + \cdots + \int_{a_0}^{b_g} \omega \pmod{\text{periods of } \omega} .$$

Iterating, we get something that looks like a group law:

$$\left(\sum_{i=1}^{g} \int_{a_0}^{a_i} \omega \right) + \left(\sum_{i=1}^{g} \int_{a_0}^{b_i} \omega \right) = \left(\sum_{i=1}^{g} \int_{a_0}^{c_i} \omega \right) \pmod{\text{periods of } \omega} ,$$

where the c_i can be expressed in terms of the a_i and b_i.

Mumford's rephrasing of Abel's theorem

If ω is any rational differential on $\mathcal{C}$, then the multi-valued function

$$a \longmapsto \int_{a_0}^{a} \omega$$

from $\mathcal{C}$ to $\mathbb{C}$ factors into a composition of three maps

$$\mathcal{C} \setminus \{\text{poles of } \omega\} \xrightarrow{\phi} J \xleftarrow{\exp} \mathrm{Lie}(J) \cong T_0(J) \xrightarrow{\ell} \mathbb{C} \ ,$$

where

- J is a commutative algebraic group,
- ℓ is linear, and
- ϕ is a morphism. Further: if $g = \dim J$, then extending to the g-fold symmetric product using the addition law on J,

$$\phi^{(g)} : (\mathcal{C} \setminus \{\text{poles of } \omega\})^{(g)} \longrightarrow J \text{ is birational .}$$

Differentials on $\mathcal{C}$ and on $\mathcal{J}_{\mathcal{C}}$

"A slightly less fancy way to put it":

For each differential ω on $\mathcal{C}$ there is

- a $\phi : \mathcal{C} \setminus \{\text{poles of } \omega\} \to J$, and
- a *translation-invariant* differential η on J

such that $\phi^*\eta = \omega$.

Hence

$$\int_{\phi(a_0)}^{\phi(a)} \eta = \int_{a_0}^{a} \omega \quad (\text{mod periods}) \; .$$

Regular differentials

Now, we restrict all of this to regular differentials (no poles: "differentials of the first kind")...

If $\mathcal{C}/\mathbb{C}$ is a nonsingular plane curve of genus g defined by

$$\mathcal{C} : F(x, y) = 0 ,$$

then its regular differentials are

$$\Omega^1(\mathcal{C}) = \left\langle \frac{x^i}{F_y(x, y)} dx \right\rangle_{i=0}^{g-1} \quad \text{where} \quad F_y := \partial F / \partial y .$$

Ex: $\mathcal{C} : y^2 = x^3 + ax + b$ has $g = 1$ and $\Omega^1(\mathcal{C}) = \langle dx/y \rangle$.

The Jacobian

"Among the ω*s, the most important are those of the 1st kind, i.e., without poles, and if we integrate all of them at once, we are led to the most important J of all: the Jacobian, which we call* $\mathcal{J}_{\mathcal{C}}$*."*

$$\mathcal{C} \xrightarrow{\phi} \mathcal{J}_{\mathcal{C}} \xleftarrow{\exp} \mathrm{Lie}(\mathcal{J}_{\mathcal{C}}) \xrightarrow{\ell} \mathbb{C} \ ,$$

We find that

- $\mathcal{J}_{\mathcal{C}}$ must be a *compact* commutative algebraic group $\implies \mathcal{J}_{\mathcal{C}}$ is a complex torus
- We have an isomorphism $\phi^* : \{\text{translation-invariant 1-forms on } \mathcal{J}_{\mathcal{C}}\} \to \Omega^1(\mathcal{C})$
- $\implies \dim \mathcal{J}_{\mathcal{C}} = \dim \Omega^1(\mathcal{C}) = g(\mathcal{C})$.

$\mathcal{J}_{\mathcal{C}}$ as a complex torus

We can write

$$\mathcal{J}_{\mathcal{C}} = V/L$$

where

- $V =$ dual of $\Omega^1(\mathcal{C})$ (a complex vector space)
- $L = \left\{\ell \in V : \ell(\omega) = \int_\gamma \omega \text{ for some 1-cycle } \gamma \text{ on } \mathcal{C}\right\}$ (ie, the lattice of $\ell \in V$ that come from periods)

...And then the map $\phi : \mathcal{C} \to \mathcal{J}_{\mathcal{C}}$ is

$$\phi(a) = \int_{a_0}^{a} \omega \pmod{L}$$

(where we can fix a path from a_0 to a.)

Since $\mathcal{J}_{\mathcal{C}}$ is a group: $V^* \cong \{\text{trans-inv. 1-forms on } \mathcal{J}_{\mathcal{C}}\} \cong \{\text{cotangent space to } \mathcal{J}_{\mathcal{C}} \text{ at any } a \in \mathcal{J}_{\mathcal{C}}\} \cong \Omega^1(\mathcal{C})$.

Algebraic construction of $\mathcal{J}_{\mathcal{C}}$

We can also construct $\mathcal{J}_{\mathcal{C}}$ algebraically.

The Riemann–Roch theorem tells us that

$$l(D) - l(K_{\mathcal{C}} - D) = \deg(D) - g + 1 ,$$

so we have a partial group law

$$\mathcal{C}^{(g)} \times \mathcal{C}^{(g)} \supset U_1 \times U_2 \to U_3 \subset \mathcal{C}^{(g)} \quad \text{with the } U_i \text{ Zariski-open} .$$

Weil showed that this can be extended into an algebraic group J with $J \supset U_4 \subset \mathcal{C}^{(g)}$ for some Zariski-open U_4.

(Remember, $C^{(g)}$ is birational to $\mathcal{J}_{\mathcal{C}}$.)

The Jacobian as the Albanese and Picard variety

The Jacobian $\mathcal{J}_{\mathcal{C}}$ is the *Albanese variety* of $\mathcal{C}$:
that is, if A is an abelian variety,
then any morphism $\psi : \mathcal{C} \to A$ factors through $\mathcal{J}_{\mathcal{C}}$.

$$\mathcal{C} \longrightarrow \mathcal{J}_{\mathcal{C}} \longrightarrow A \ .$$

The Jacobian is also isomorphic to the Picard variety $\mathrm{Pic}^0(\mathcal{C})$ of $\mathcal{C}$ via the Abel–Jacobi theorem.

(Note: $\mathrm{Pic}^0(\mathcal{X})$ is the dual of $\mathrm{Alb}(\mathcal{X})$.)

Abel–Jacobi

Given $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ in $\mathcal{C}$,

$$\sum_{i=1}^{k} \phi(x_i) = \sum_{i=1}^{k} \phi(y_i) \iff \sum_{i=1}^{k} x_i - \sum_{i=1}^{k} y_i = (f) \text{ for some } f \in \mathbb{C}(\mathcal{C}) \ .$$

Consider the map $\phi^{(k)} : \mathcal{C}^{(k)} \to \mathcal{J}_{\mathcal{C}}$; we define subvarieties

$$W_k := \mathrm{Image}(\phi^{(k)}) \subseteq \mathcal{J}_{\mathcal{C}} \quad \text{for } k \geq 1$$

(so $W_k = \mathcal{J}_{\mathcal{C}}$ if $k \geq g(\mathcal{C})$).

The most important of these is the Theta divisor

$$W_{g-1} =: \Theta \ .$$

Θ is ample; the functions in $L(n\Theta)$ map $\mathcal{J}_{\mathcal{C}}$ into $\mathbb{P}^{n^g-1}$.

Fibres of $\phi^{(k)}$

Abel's theorem $\implies$ the fibres of $\phi^{(k)}$ are *linear systems of degree k*, hence $\cong$ projective spaces:

- Pick a degree-k effective divisor $D \in \mathcal{C}^{(k)}$.
- Riemann–Roch space $L(D) := \{f \in \mathbb{C}(\mathcal{C}) : (f) + D \geq 0\}$.
- $|D| := \{(f) + D : f \in L(D)\} = (\phi^{(k)})^{-1}(\phi^{(k)}(D)) \cong \mathbb{P}(L(D))$ is the linear system of effective divisors linearly equivalent to D

Hence: if $x = \phi^{(k)}(D)$, then $(\phi^{(k)})^{-1}(x) = |D| \cong \mathbb{P}(L(D))$.
Riemann–Roch $\implies$ $\dim |D| = k - g + \dim \Omega^1(-D)$
where $\Omega^1(-D) =$ space of ω in $\Omega^1(\mathcal{C})$ with zeroes on D.

Consequence:

- $\phi^{(1)}(\mathcal{C}) = \mathrm{pt} \iff g(\mathcal{C}) = 0 \iff \mathcal{C} \cong \mathbb{P}^1$
- $\phi^{(1)}$ is an embedding $\mathcal{C} \xrightarrow{\sim} W_1 \subseteq \mathcal{J}_{\mathcal{C}} \iff g(\mathcal{C}) \geq 1$.

Genus 0

If $\mathcal{C}$ has genus zero, then $\Omega^1(\mathcal{C}) = 0$, so

$$\mathcal{J}_{\mathcal{C}} = 0$$

...which fits with Riemann–Roch: $\mathcal{J}_{\mathcal{C}} \cong \mathrm{Pic}^0(\mathcal{C}) = 0$.

Since $\mathcal{J}_{\mathcal{C}} = \mathrm{Alb}(\mathcal{C}) = 0$, we find that for any curve $\mathcal{X}$,

- The only linear subvarieties of $\mathcal{J}_{\mathcal{X}}$ are points (lines L mapping into $\mathcal{J}_{\mathcal{X}}$ map through $\mathrm{Alb}(L) = 0$).
- More generally, there are no rational curves in any $\mathcal{J}_{\mathcal{X}}$.

Genus 1

If $\mathcal{C}$ has genus one, then $\phi : \mathcal{C} \to \mathcal{J}_{\mathcal{C}}$ is an embedding, hence

$$\mathcal{J}_{\mathcal{C}} \cong \mathcal{C} \ .$$

(The isomorphism depends on ϕ, ie on the choice of base point a_0.)

In terms of the Picard group: the isomorphism $\mathcal{C} \to \mathcal{J}_{\mathcal{C}} \cong \mathrm{Pic}^0(\mathcal{C})$ is defined by $a \mapsto [a - a_0]$, and $[a - a_0] + [b - b_0] = [(a \oplus b) - a_0]$.

$$\int_{a_0}^{a} dx/y + \int_{a_0}^{b} dx/y = \int_{a_0}^{a} dx/y + \int_{a_0 \oplus a}^{b \oplus a} dx/y = \int_{a_0}^{a \oplus b} dx/y \quad \text{(mod periods)}$$

In this case, $\Theta = W_0 = a_0$ (so a_0 "is" the principal polarization).

Indeed, $|3\Theta|$ defines a projective embedding of $\mathcal{J}_{\mathcal{C}}$ into $\mathbb{P}^2$.

Genus 2

Let $\mathcal{C}$ be a curve of genus 2, and consider

$$\phi^{(2)} : \mathcal{C}^{(2)} \longrightarrow \mathcal{J}_{\mathcal{C}} \ .$$

The preimage of each point of $\mathcal{J}_{\mathcal{C}}$ is either a point or a line.

$\mathcal{C} : y^2 = f(x)$ has a hyperelliptic $\pi : \mathcal{C} \xrightarrow{2} \mathbb{P}^1$ mapping $(x, y) \mapsto x$. All points of $\mathbb{P}^1$ are linearly equivalent $\implies$ all of the $\pi^{-1}(x)$ are linearly equivalent, so we get a copy of $\mathbb{P}^1$ in $\mathcal{C}^{(2)}$:

$$E = \{(x, y) + (x, -y) : x \in \mathbb{P}^1\} \subset \mathcal{C}^{(2)} \ .$$

Result: $\mathcal{J}_{\mathcal{C}}$ is obtained from $\mathcal{C}^{(2)}$ by "blowing down" the divisor $E \cong \mathbb{P}^1$ to a single point.

In this case: $\Theta = \phi(\mathcal{C})$ is a copy of $\mathcal{C}$ inside $\mathcal{J}_{\mathcal{C}}$.

Genus 3

Let $\mathcal{C}$ be a curve of genus 3.

First, consider $k = 3$:

$$\phi^{(3)} : \mathcal{C}^{(3)} \to \mathcal{J}_{\mathcal{C}} \ .$$

Fix x in $\mathcal{C}$; the $\omega \in \Omega^1(\mathcal{C})$ zero at x form a 2-dimensional space. These ω have 3 other zeroes $\to$ determine degree-3 effective divisors, and these effective divisors are linearly equivalent (via $f = \omega_1/\omega_2$), and hence form a linear system:

$$\text{each } x \in \mathcal{C} \longleftrightarrow \text{a copy } E_x \text{ of } \mathbb{P}^1 \text{ in } \mathcal{C}^{(3)} \ .$$

Now we get $\mathcal{J}_{\mathcal{C}}$ from $\mathcal{C}^{(3)}$ by blowing down each E_x to a point.

On the other hand: if $\gamma = \{\phi^{(3)}(E_x) : x \in \mathcal{C}\} \subset \mathcal{J}_{\mathcal{C}}$, then $\gamma \cong \mathcal{C}$, and $\mathcal{C}^{(3)} = \mathcal{J}_{\mathcal{C}}$ blown up along γ.

Genus 3, continued

Next, consider $k = 2$ (still with $g(\mathcal{C}) = 3$):

$$\phi^{(2)} : \mathcal{C}^{(2)} \to W_2 \subset \mathcal{J}_{\mathcal{C}} \ .$$

If $\mathcal{C}$ is *nonhyperelliptic* then there are no nontrivial degree-2 linear systems, so no preimages under $\phi^{(2)}$ of dimension > 0, so

$$W_2 \cong \mathcal{C}^{(2)} \ .$$

If $\mathcal{C}$ is *hyperelliptic*: one degree-2 linear system E (from the hyperelliptic $C \to \mathbb{P}^1$, like in $g = 2$), and so

$$\Theta = W_2 \cong \mathcal{C}^2 \text{ with } E \text{ blown down to a point} \ .$$

The image $e = \phi^{(2)}(E)$ of E in W_2 is a double point.

Genus 4 : Hyperelliptic case

Let $\mathcal{C}$ be a curve of genus 4.

First, if $\mathcal{C}$ is hyperelliptic:
$\exists$ a degree-2 linear system E from the hyperelliptic $\pi : \mathcal{C} \xrightarrow{2} \mathbb{P}^1$.

Hence for each $x \in \mathcal{C}$ we have a degree-3 linear system

$$\mathbb{P}^1 \cong E_x = E + x \subset \mathcal{C}^3$$

—ie $\mathcal{C}^{(3)}$ contains a whole curve of $\mathbb{P}^1$s.

Let S be the surface $\cup_{x\in\mathcal{C}} E_x \subset \mathcal{C}^{(3)}$. Then we have

$$\Theta = W_3 \cong \mathcal{C}^{(3)} \text{ with } S \text{ blown down to a curve } \gamma \cong \mathcal{C} \ ,$$

and γ is a double curve of W_3.

Genus 4 : Nonhyperelliptic general case

Suppose $\mathcal{C}$ is nonhyperelliptic of genus 4.
Then $\mathcal{C}$ is the intersection of a quadric F and a cubic G in $\mathbb{P}^3$.

General case: $F \cong \mathbb{P}^1 \times \mathbb{P}^1$.
$\implies$ two projections $\pi_i : \mathcal{C} \to \mathbb{P}^1$ of degree 3
$\implies$ two linear systems $E_1, E_2 \subset \mathcal{C}^{(3)}$, with $E_1 \cong E_2 \cong \mathbb{P}^1$.

Here:

$$\Theta = W_3 \cong \mathcal{C}^{(3)} \text{ with } E_1, E_2 \text{ blown down to points } e_1, e_2$$

and e_1 and e_2 are ordinary double points of W_3.

Genus 4 : Nonhyperelliptic general case

Suppose $\mathcal{C}$ is nonhyperelliptic of genus 4.
Then $\mathcal{C}$ is the intersection of a quadric F and a cubic G in $\mathbb{P}^3$.

Special case: F is a singular quadric. Then the two degree-3 maps $\mathcal{C} \to \mathbb{P}^1$ coincide, so there is only one nontrivial degree-3 linear system, E:

$$\Theta = W_3 \cong \mathcal{C}^{(3)} \text{ with } E \text{ blown down to } e$$

and e is a higher double point of W_3.